Postcritical reflection When $\beta_2 > \beta_1$, ϕ_2 can be 90° and ϕ_1 in this condition is called *critical angle*:

$$\phi_c = \sin^{-1}\frac{\beta_1}{\beta_2}. \tag{2.101}$$

When the incident angle is larger than ϕ_c, we have postcritical reflection, in which waves are perfectly reflected. In this case, $\eta_2 = \sqrt{\beta_2^2 - p^2}$ is imaginary. To avoid divergence of refracted waves of v_2 (equation 2.93) at $z \to +\infty$, the sign of η_2 should be

$$\eta_2 = i\hat{\eta}_2 = i\sqrt{p^2 - \beta_2^{-2}}\ (\omega > 0) \tag{2.102}$$

When medium 2 has a finite thickness (H) and the free surface exists on top of it, waves reverberate. The solution in medium 1 is the same as equation equation 2.93. Because we have another reflected waves from the boundary at $z = H$, the solution in medium 2 is

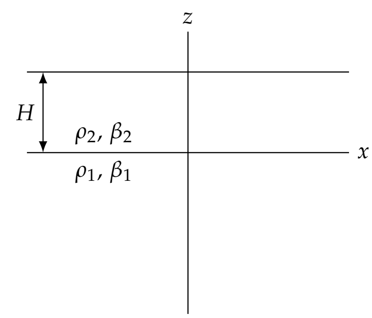

Figure 2.9: Reflection and transmission at a medium which has the free surface and a finite layer.

$$v_2 = A_2 e^{-i\omega(t-px-\eta_2(z-H))} + B_2 e^{-i\omega(t-px+\eta_2(z-H))}. \tag{2.103}$$

Because the stress σ_{yz} is 0 at the free surface $z = H$, we obtain $A_2 = B_2$. Therefore, equation 2.103 becomes

$$v_2 = 2A_2 \cos\omega\eta_2(z-H)e^{-i\omega(t-px)}. \tag{2.104}$$

The boundary condition at $z = 0$ is the same as equation 2.95 and we obtain

$$\begin{aligned} A_1 + B_1 &= 2A_2\cos\omega\eta_2 H \\ i\mu_1\eta_1(A_1 - B_1) &= 2\mu_2\eta_2 A_2 \sin\omega\eta_2 H. \end{aligned} \tag{2.105}$$

From equation 2.105, we can compute reflection and transmission coefficients:

Different from equations 2.92 or 2.97, equation 2.106 is a function of the frequency. This is because the reflection and transmission depend on the thickness H.
Proof $|R| = 1$.

$$\begin{aligned} T &= \frac{A_2}{A_1} = \frac{\mu_1\eta_1}{\mu_1\eta_1\cos\omega\eta_2 H - i\mu_2\eta_2\sin\omega\eta_2 H}, \\ R &= \frac{B_1}{A_1} = \frac{\mu_1\eta_1\cos\omega\eta_2 H + i\mu_2\eta_2\sin\omega\eta_2 H}{\mu_1\eta_1\cos\omega\eta_2 H - i\mu_2\eta_2\sin\omega\eta_2 H}. \end{aligned} \tag{2.106}$$

Waves are amplified because of the surface layer. The amplitude ratio between the incident wave and the wave represented by equation 2.103 is

$$\left|\frac{v_2(z=H)}{A_1}\right| = \left|\frac{2A_2}{A_1}\right| = 2|T|. \tag{2.107}$$

Compared with the ratio without the surface layer (2 due to equation 2.92), $|T|$ relates to the amplification of the waves.

If η_i is real, the denominator of T is following an ellipse on the real-imaginary domain with principal axes on the real and imaginary axes when ω changes. Therefore, the maximum and minimum T should be on the real or imaginary axes. On the real axis ($\sin\omega\eta_2 H = 0$ and $\cos\omega\eta_2 H = \pm 1$),

$$|T| = 1, \tag{2.108}$$

and on the imaginary axis ($\sin\omega\eta_2 H = \pm 1$ and $\cos\omega\eta_2 H = 0$),

$$|T| = \frac{\mu_1\eta_1}{\mu_2\eta_2} = \frac{\rho_1\beta_1\cos\phi_1}{\rho_2\beta_2\cos\phi_2}. \tag{2.109}$$

When we consider the vertical incident wave ($\phi_1 = \phi_2 = 0$), the maximum $|T|$ is on the real axis (equation 2.108) when the surface layer is harder than below ($\rho_1\beta_1 < \rho_2\beta_2$). On the other hand, when the surface layer is softer ($\rho_1\beta_1 > \rho_2\beta_2$), the maximum $|T|$ is on the imaginary axis (equation 2.109) and $|T| > 1$, which is the reason of amplification at the soft structure (e.g., figure 2.10). The frequency at the maximum amplification satisfies $\cos\omega\eta_2 H = 0 \to \omega\eta_2 H = (2n+1)\pi/2$.

The T and R (equation 2.106) include all reverberations (p101-102, Saito).

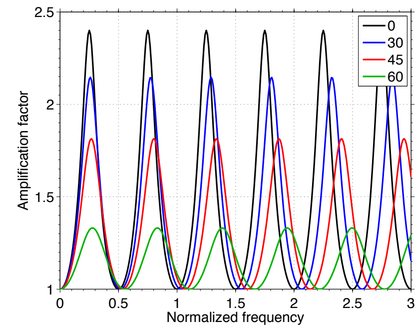

Figure 2.10: Site amplification caused by a soft surface layer for SH waves for different incident angles (line colors). The normalized frequency is fH/β_2 and the vertical axis $|T|$. In this example, I use $\rho_1/\rho_2 = 1.2$ and $\beta_1/\beta_2 = 2$.

2.6.4 P-SV waves

2.7 *Surface waves*

Surface and body waves are not very easy to distinguish because they are related. We consider that surface waves are propagating around the surface of media and the energy of them concentrate near the surface. Generally, the main features of surface waves compared with body waves are traveling slower, less amplitude decay, and velocities are frequency dependent.

2.7.1 *Dispersion*

One important feature is that surface waves are dispersive (in contrast to body waves), which means that the depth sensitivity of surface waves depends on frequencies of waves, and hence we can obtain vertical heterogeneity of subsurface from surface waves.

The simplest example of dispersion may be the sum of two harmonic waves with slightly different frequency and wavenumber (Figure 2.11):

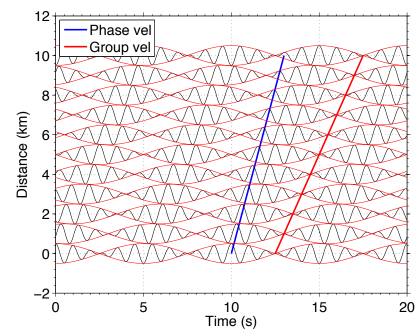

Figure 2.11: Superimposed cosine waves. Here, $\omega = 1 \times 2 \times \pi$ (1/s), $k = 0.3 \times 2 \times \pi$ (1/km), $\delta\omega = 0.1$ (1/s), and $\delta k = 0.05$ (1/km).

$$u(x,t) = \cos(\omega_1 t - k_1 x) + \cos(\omega_2 t - k_2 x), \tag{2.110}$$

where $\omega_1 = \omega - \delta\omega$, $\omega_2 = \omega + \delta\omega$, $k_1 = k - \delta k$, and $k_2 = k + \delta k$. Therefore,

$$\begin{aligned} u(x,t) &= \cos\{(\omega t - kx) - (\delta\omega t - \delta kx)\} + \cos\{(\omega t - kx) + (\delta\omega t - \delta kx)\} \\ &= 2\cos(\omega t - kx)\cos(\delta\omega t - \delta kx). \end{aligned} \tag{2.111}$$

$\cos(a+b) + \cos(a-b) = 2\cos a \cos b$

The waveform of $u(x,t)$ consists of a cosine curve with frequency ω (*carrier*) with a superimposed cosine curve with frequency $\delta\omega$ (*envelope*). From equation 2.111, the velocities for short (carrier) and long (envelope) period waves are

$$c = \frac{\omega}{k},\ U = \frac{d\omega}{dk}, \tag{2.112}$$

respectively. In equation 2.112, we assume $\delta\omega$ and δk approach to zero. We call c as *phase velocity* and U as *group velocity*. The group velocity U can be written as

$$U = \frac{d\omega}{dk} = c + k\frac{dc}{dk} = c\left(1 - k\frac{dc}{d\omega}\right)^{-1}. \tag{2.113}$$

$$\begin{aligned} d\omega &= \omega - \omega_1 = ck - c_1k_1 = ck - (c - dc)(k - dk) \\ &\approx cdk + kdc \\ dk &= k - k_1 = \frac{\omega}{c} - \frac{\omega_1}{c_1} = \frac{\omega}{c} - \frac{\omega - d\omega}{c - dc} \\ &\approx \frac{\omega}{c} - \frac{\omega - d\omega}{c} - \frac{\omega dc - dcd\omega}{c^2} \approx \frac{d\omega}{c} - \frac{\omega dc}{c^2} \\ \frac{1}{U} &= \frac{dk}{d\omega} = \frac{d\omega/c - \omega dc/c^2}{d\omega} = \frac{1}{c}\left(1 - k\frac{dc}{d\omega}\right) \end{aligned}$$

Usually, because the phase velocity c of Love and Rayleigh waves increase with period (i.e., velocity increasing with depth), $dc/d\omega$ is negative. Therefore, the group velocity is slower than the phase velocity $U < c$.

2.7.2 *Love waves*

We consider the medium shown in Figure 2.12, which contains a finite thickness layer on top of a halfspace medium. Note that we need a layer to obtain Love waves. The Love-wave problem can be considered as that whether waves, which horizontally propagate with velocity c and amplitude zero at $z \to \infty$, exist or not.

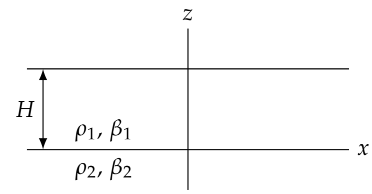

Figure 2.12: Two-layer model. I should follow the subscripts with Figure 2.9.

Love waves within a homogeneous layer can result from constructive interference between postcritical reflected SH waves.

$$\eta_1 = \sqrt{\beta_1^{-2} - p^2},\ c = 1/p$$

When we consider the condition $\beta_1 < c < \beta_2$ (which is the condition that Love waves exist I will proof later.), a solution in the medium 1 is

$$v_1(z) = \cos \omega\eta_1(z - H)e^{-i\omega(t-px)}, \tag{2.114}$$

which is equal to equation 2.104 with $A = 1/2$. Based on equation 2.93, a solution in the medium 2 is

$$v_2 = A_2 e^{-i\omega(t-px-\eta_2 z)} + B_2 e^{-i\omega(t-px+\eta_2 z)}, \tag{2.115}$$

where $\eta_2^2 < 0$ when $c < \beta_2$. When we choose $\Im(\eta_2) > 0$ $(\omega > 0)$, the first and second terms on the right-hand side of equation 2.115 are diverse and converse to zero at $z \to -\infty$, respectively. By considering the condition of amplitudes, we can write a solution in the medium 2 as

$$e^{i\omega\eta_2 z} = e^{i\omega(\Re(\eta_2)+i\Im(\eta_2))z} = \underbrace{e^{i\omega\Re(\eta_2)z}}_{oscillation} \underbrace{e^{-\omega\Im(\eta_2)z}}_{divergence(z=\infty)}$$

Because η_2 is complex number, the reflected waves from the medium 1 perfectly reflect at the boundary $z = 0$. Also from equation 2.121,

$$v_1 = e^{-i\omega(t-px)}\left[e^{i\omega\eta_1(z-H)} + e^{-i\omega\eta_1(z-H)}\right],$$

which is the summation of upgoing and downgoing plane waves (with propagating to the $+x$ direction. Therefore, we can consider Love waves are reverberation of SH waves.

$$v_2 = B_2 e^{-i\omega(t-px+\eta_2 z)} = B_2 e^{-i\omega(t-px)} e^{\omega\hat{\eta}_2 z}, \tag{2.116}$$

where $\hat{\eta}_2 = \sqrt{p^2 - \beta_2^{-2}} > 0$.

Because the boundary condition at the free surface is already satisfied in equation 2.114, the boundary condition at $z = 0$ should be satisfied (displacements and stresses should be continuous):

$$v_1 = v_2,\ \mu_1 \frac{\partial v_1}{\partial z} = \mu_2 \frac{\partial v_2}{\partial z}$$
$$\cos \omega\eta_1 H = B_2,\ \mu_1(\omega\eta_1 \sin \omega\eta_1 H) = \mu_2(\omega\hat{\eta}_2 B_2) \tag{2.117}$$

where $\mu_i = \rho_i \beta_i^2$. Therefore, to exist Love waves, waves satisfy

$$\Delta_l(p, \omega) = \mu_2\hat{\eta}_2 \cos \omega\eta_1 H - \mu_1\eta_1 \sin \omega\eta_1 H = 0, \tag{2.118}$$

or

$$\tan \omega\eta_1 H = \frac{\mu_2\hat{\eta}_2}{\mu_1\eta_1}, \tag{2.119}$$

which are called the characteristic equation for Love waves. With equation 2.120, Love waves exist when η_1 and $\hat{\eta}_2$ are real positive number for an angular frequency.

Mode The equation defines the dispersion curve for Love wave propagation within the layer. On the plane of $p\omega$, for each p, we have multiple values of ω satisfies equation 2.120 due to the tangent function, and the smallest ω defines the fundamental mode, and the second smallest is the first higher mode, etc. Equation 2.120 cannot be solved analytically, but we can do numerically. When ω is small, we only have one solution, which is the fundamental mode (Saito, p149). Also in the fundamental mode, $c \to \beta_2$ ($\omega \to 0$) and $c \to \beta_1$ ($\omega \to \infty$).

The angular frequency of nth higher modes can be defined as

$$\frac{\omega_n H}{\beta_1} = \frac{n\pi}{\sqrt{1-(\beta_1/\beta_2)^2}}, \tag{2.120}$$

and called *cut-off angular frequency*.

Depth variation of amplitude From equations 2.114, 2.116, and 2.117, the displacements of Love waves are

$$\begin{aligned} v_1(z) &= \underbrace{\cos \omega\eta_1(z-H)}_{amplitude}\underbrace{e^{-i\omega(t-px)}}_{phase} \\ v_2(z) &= \underbrace{\cos \omega\eta_1(H)e^{\omega\hat{\eta}_2 z}}_{amplitude}\underbrace{e^{-i\omega(t-px)}}_{phase}. \end{aligned} \tag{2.121}$$

Group velocity We can estimate the group velocity of Love waves by computing equation 2.113. The $p(\omega)$ derivative of $\Delta_L(p,\omega) = 0$ is

$$\begin{aligned} \frac{\partial\Delta_L(p,\omega)}{\partial\omega} + \frac{\partial\Delta_L(p,\omega)}{\partial p}\frac{\partial p(\omega)}{\partial\omega} &= 0 \\ \frac{\partial p(\omega)}{\partial\omega} &= -\frac{\partial\Delta_L/\partial\omega}{\partial\Delta_L/\partial p} \end{aligned} \tag{2.122}$$

When $f(x,y)=0$,

$$\begin{aligned} \frac{d}{dx}f(x,y(x)) &= 0 \\ \frac{df(x,y)}{dx} + \frac{df(x,y)}{dy}\frac{dy(x)}{dx} &= 0 \end{aligned}$$

For the two-layer case (equation 2.118),

$$\frac{c}{U} = 1 + \frac{\eta_1^2}{p^2}\left[1 + \frac{(\mu_2/\mu_1)(\beta_1^{-2}-\beta_2^{-2})}{\omega\hat{\eta}_2 H[\eta_1^2 + (\mu_2/\mu_1)^2\hat{\eta}_2^2]}\right]^{-1}$$

2.7.3 *Rayleigh waves*