1.1 *Stress, strain, and displacement → wave equation*

From the relationship between stress, strain, and displacement, we can derive a 3D elastic wave equation. Figure 1.1 shows relationships between each pair of parameters. In this section, I will show each term in Figure 1.1.

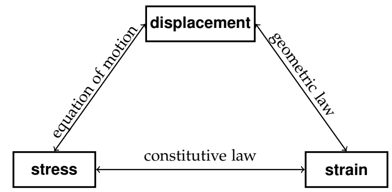

Figure 1.1: Relationship of each parameter.

1.1.1 *Displacement*

Displacement, characterizes vibrations, is distance of a particle from its position of equilibrium:

$$\mathbf{u}(\mathbf{x},t) = \begin{pmatrix} u_1(\mathbf{x},t) \\ u_2(\mathbf{x},t) \\ u_3(\mathbf{x},t) \end{pmatrix}. \tag{1.1}$$

1.1.2 *Stress*

Stress characterizes forces applied to a material:

$$\sigma_{ij} = \underline{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}, \tag{1.2}$$

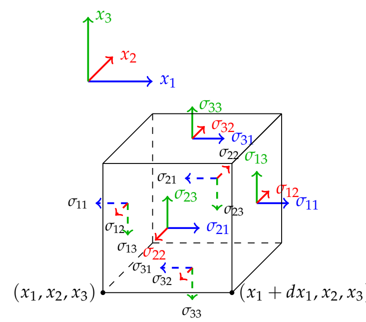

Figure 1.2: Stresses.

which is a tensor, and the first subscript indicates the surface applied and the second the direction (Figure 1.2).

1.1.3 *Strain*

Strain characterizes deformations under stress. If stresses are applied to a material that is not perfectly rigid, points within it move with respect to each other, and deformation results.

Let us consider an elastic material which moves $\mathbf{u}(\mathbf{x})$ (Figure 1.3). When the original location of the material is $\mathbf{x}$, the displacement of a nearby point originally at $\mathbf{x} + \delta\mathbf{x}$ can be written as

$$u_i(\mathbf{x}+\delta\mathbf{x}) \approx u_i(\mathbf{x}) + \frac{\partial u_i(\mathbf{x})}{\partial x_j}\delta x_j = \underbrace{u_i(\mathbf{x})}_{parallel\ translation} + \underbrace{\delta u_i}_{rotation+deformation}, \tag{1.3}$$

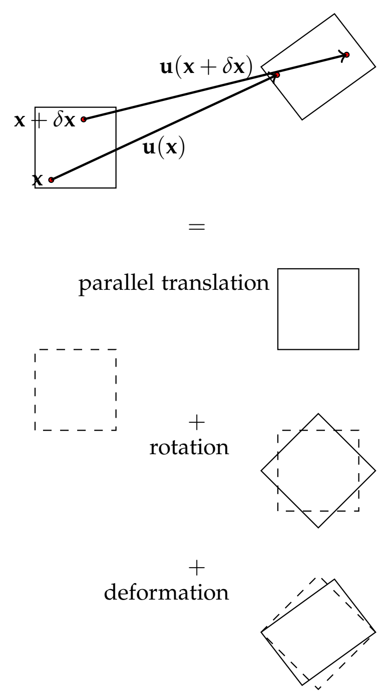

Figure 1.3: Displacement includes parallel translation, rotation, and deformation (strain).

Therefore, in the first-order assumption,

$$\begin{aligned}
\delta u_i &= \frac{\partial u_i(\mathbf{x})}{\partial x_j}\delta x_j \\
&= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\delta x_j + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\delta x_j \\
&= \frac{1}{2}(u_{i,j} + u_{j,i})\delta x_j + \frac{1}{2}(\nabla \times \mathbf{u} \times \delta \mathbf{x})_i \\
&= (e_{ij} + \omega_{ij})\delta x_j,
\end{aligned} \tag{1.4}$$

where ω_{ij} is a rotational translation term (diagonal term is zero, $\omega_{ij} = -\omega_{ji}$). Then $e_{ij} = \underline{\mathbf{e}}$ is the strain tensor, which contains the spatial derivatives of the displacement field. With the definition of e_{ij}, the tensor is symmetric and has 6 independent components.

$$e_{ij} = \begin{pmatrix} u_{1,1} & 1/2(u_{1,2} + u_{2,1}) & 1/2(u_{1,3} + u_{3,1}) \\ 1/2(u_{2,1} + u_{1,2}) & u_{2,2} & 1/2(u_{2,3} + u_{3,2}) \\ 1/2(u_{3,1} + u_{1,3}) & 1/2(u_{3,2} + u_{2,3}) & u_{3,3} \end{pmatrix} \tag{1.5}$$

If the diagonal terms of e_{ij} are zero, we do not have volume changes. The volume increase, *dilatation*, is given by the sum of the extensions in the x_i directions:

$$e_{ii} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = tr(\mathbf{e}) = \nabla \cdot \mathbf{u} = \theta \tag{1.6}$$

This dilatation gives the change in volume per unit volume associated with the deformation. $\partial u_i / \partial x_i$ mentions displacement of the x_i direction changes along the direction of x_i.

$$\left(1 + \frac{\partial u_1}{\partial x_1}\right) dx_1 \left(1 + \frac{\partial u_2}{\partial x_2}\right) dx_2 \left(1 + \frac{\partial u_3}{\partial x_3}\right) dx_3 \approx \left(1 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) dx_1 dx_2 dx_3 = (1+\theta)V = V + \Delta V, \tag{1.7}$$

where $\theta = \Delta V / V$.

1.1.4 Geometric law

Relationship between displacement and strain, which represents geometric properties (deformation).

As we have already found in equation 1.4,

$$\underline{\mathbf{e}} = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right) \tag{1.8}$$

1.1.5 Equation of motion

Relationship between displacement and stress, which represents dynamic properties (motion).

We write Newton's second law in terms of body forces and stresses. When I consider the stresses in the x_2 direction (the red arrows in Figure 1.2),

$$\begin{aligned} &\{\sigma_{12}(\mathbf{x}+dx_1\hat{\mathbf{n}}_1)-\sigma_{12}(\mathbf{x})\}\,dx_2dx_3 \\ &+\{\sigma_{22}(\mathbf{x}+dx_2\hat{\mathbf{n}}_2)-\sigma_{22}(\mathbf{x})\}\,dx_1dx_3 \\ &+\{\sigma_{32}(\mathbf{x}+dx_3\hat{\mathbf{n}}_3)-\sigma_{32}(\mathbf{x})\}\,dx_1dx_2 \\ &\qquad\qquad +f_2dV=\rho\frac{\partial^2u_2}{\partial t^2}dV \end{aligned} \tag{1.9}$$

where $dV = dx_1dx_2dx_3$. With a Taylor expansion,

$$\left(\frac{\partial\sigma_{12}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+\frac{\partial\sigma_{32}}{\partial x_3}\right)dV+f_2dV=\rho\frac{\partial^2u_2}{\partial t^2}dV \tag{1.10}$$

We also have similar equations for x_1 and x_2 directions, and by using the summation convention,

$$\begin{aligned} \underbrace{\sigma_{ij,j}(\mathbf{x},t)}_{surface\ forces}+\underbrace{f_i(\mathbf{x},t)}_{body\ forces}&=\rho\frac{\partial^2u_i(\mathbf{x},t)}{\partial t^2} \\ \nabla\cdot\underline{\boldsymbol{\sigma}}+\mathbf{f}&=\rho\ddot{\mathbf{u}}. \end{aligned} \tag{1.11}$$

This is the equation of motion, which is satisfied everywhere in a continuous medium. When the right-hand side in equation 1.11 is zero, we have the equation of equilibrium,

$$\sigma_{ij,j}(\mathbf{x},t)=-f_i(\mathbf{x},t), \tag{1.12}$$

and if no body forces are applied, we have the homogeneous equation of motion

$$\sigma_{ij,j}(\mathbf{x},t)=\rho\frac{\partial^2u_i(\mathbf{x},t)}{\partial t^2}. \tag{1.13}$$

1.1.6 Constitutive equations

Relationship between stress and strain, which represents material properties (strength, stiffness). Here, we consider the material has a linear relationship between stress and strain (linear elastic). Linear elasticity is valid for the short time scale involved in the propagation of seismic waves.

Based on Hooke's law, the relationship between stress and strain is

$$\begin{aligned} \sigma_{ij}&=c_{ijkl}e_{kl} \\ \underline{\boldsymbol{\sigma}}&=\underline{\underline{\mathbf{c}}}\ \underline{\mathbf{e}}, \end{aligned} \tag{1.14}$$

where constant c_{ijkl} is the elastic moduli, which describes the properties of the material.

Not all components of c_{ijkl} are independent. Because stress and strain tensors are symmetric and thermodynamic consideration;

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}. \tag{1.15}$$

Therefore, we have 21 independent components in c_{ijkl}. With Voigt recipe, we change the subscripts with

$$11 \to 1,\ 22 \to 2,\ 33 \to 3,\ 23 \to 4,\ 13 \to 5,\ 12 \to 6,$$

and we can write the elastic moduli as c_{ij} $(i, j = 1, 2, \cdots, 6)$. With these 21 components, we can describe general anisotropic media.

Strain energy is defined by

$$W = \frac{1}{2} \int \sigma_{ij} e_{ij} dV = \frac{1}{2} \int c_{ijkl} e_{ij} e_{kl} dV,$$

Therefore, $c_{ijkl} = c_{klij}$.

1.1.7 Wave equation (general anisotropic media)

Wave equation describes vibrations ($\mathbf{u}$) at each space ($\mathbf{x}$) and time (t) under material properties ($\underline{\underline{\mathbf{c}}}$, ρ);

$$f(\mathbf{u}, \mathbf{x}, t, \rho, \underline{\underline{\mathbf{c}}}) = \mathbf{F}. \tag{1.16}$$

In homogeneous case ($\mathbf{F} = 0$),

$$f(\mathbf{u}, \mathbf{x}, t, \rho, \underline{\underline{\mathbf{c}}}) = 0. \tag{1.17}$$

We eliminate $\underline{\boldsymbol{\sigma}}$ and $\underline{\mathbf{e}}$ by plugging in equations 1.8, 1.11, and 1.14.

$$\nabla \cdot \left\{ \underline{\underline{\mathbf{c}}} \left(\frac{1}{2} \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right] \right) \right\} = \rho \ddot{\mathbf{u}} \tag{1.18}$$

This is a general wave equation for anisotropic elastic media.

geometric law $\underline{\mathbf{e}} = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)$ (eq 1.8)

- small perturbation

equation of motion $\nabla \cdot \underline{\boldsymbol{\sigma}} = \rho \ddot{\mathbf{u}}$ (eq 1.11)

- small perturbation
- continuous material

constitutive law $\underline{\boldsymbol{\sigma}} = \underline{\underline{\mathbf{c}}}\ \underline{\mathbf{e}}$ (eq 1.14)

- small perturbation
- continuous material
- elastic material

1.1.8 Elastic moduli in isotropic media

On a large scale (compared with wave length), the earth has approximately the same physical properties regardless of orientation, which is called *isotropic*. In the isotropic case, c_{ijkl} has only two independent components. One pair of the components are called the Lamé constants λ and μ, which are defined as

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \tag{1.19}$$

μ is called the shear modulus, but λ does not have clear physical explanation. By using the Voigt recipe, equation 1.18 can be written with a matrix form;

$$c_{ij} = \begin{pmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix} \tag{1.20}$$

In the isotropic media, equation 1.14 becomes

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij}$$
$$\underline{\sigma} = \lambda tr(\underline{\mathbf{e}})\mathbf{I} + 2\mu\underline{\mathbf{e}} \tag{1.21}$$

where θ is the dilatation.

There are other elastic moduli, which are related to the Lamé constants, such as bulk modulus (K), Poisson's ratio (ν), and Young's modulus (E) (Table 1.1.8).

Table 1.1: Elastic moduli

	(λ,μ)	(λ,ν)	(K,λ)	(E,μ)	(K,μ)	(E,ν)	(μ,ν)	(K,ν)	(K,E)
K	$\lambda + \frac{2}{3}\mu$	$\frac{\lambda(1+\nu)}{3\nu}$		$\frac{E\mu}{3(3\mu-E)}$		$\frac{E}{3(1-2\nu)}$	$\frac{2\mu(1+\nu)}{3(1-2\nu)}$		
ν	$\frac{\lambda}{2(\lambda+\mu)}$		$\frac{\lambda}{3K-\lambda}$	$\frac{E}{2\mu}$	$\frac{3K-2\mu}{2(3K+\mu)}$				$\frac{3K-E}{6K}$
E	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	$\frac{9K(K-\lambda)}{3K-\lambda}$		$\frac{9K\mu}{3K+2\mu}$		$2\mu(1+\nu)$	$3K(1-2\nu)$	
λ									
μ									

1.1.9 *Wave equation in isotropic media*

Using equation 1.21 instead of equation 1.14, we can derive the wave equation in an isotropic medium.

From equations 1.8, 1.11, and 1.21, the isotropic wave equation is

$$\rho\ddot{\mathbf{u}} = (\lambda + 2\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\nabla\times\nabla\times\mathbf{u}, \tag{1.22}$$

with an assumption of slowly-varying material ($\nabla\lambda \approx 0$ and $\nabla\mu \approx 0$).

geometric law $\underline{\mathbf{e}} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$ (eq 1.8)

equation of motion $\nabla\cdot\underline{\sigma} = \rho\ddot{\mathbf{u}}$ (eq 1.11)

constitutive law $\underline{\sigma} = \lambda tr(\underline{\mathbf{e}})\mathbf{I} + 2\mu\underline{\mathbf{e}}$ (eq 1.21)

$\nabla\cdot\mathbf{u}$ volumetric deformation

$\nabla\times\mathbf{u}$ shearing deformation